The Riemann Hypothesis for Period Polynomials of Modular and Hilbert Modular Forms

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The Riemann Hypothesis for Period Polynomials (RHPP) is the assertion that all the roots of period polynomials of modular forms lie on a circle centered at the origin.

- Conrey, Farmer and Imamoglu (2013): the odd part of the period polynomial for any level 1 cusp form has roots on the unit circle.
- El-Guindy and Raji (2014): extend to the full polynomial
- Jin, Ma, Ono and Soundararajan (2016): generalized the result for modular forms of higher levels
- Diamantis and Rolen (2018): conjecture for period polynomials associated to higher derivatives of L-functions
- Babei, Rolen and Wagner (2021): analogous result for Hilbert modular forms on the full Hilbert modular group.

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$SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ in the standard way by *Möbius* transformations:

$$\text{For } z \in \mathbb{C} \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ \gamma.z = \frac{az + b}{cz + d}$$

Definition

A modular form of weight $k \in \mathbb{Z}$ on Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

- $f(\gamma z) = (cz+d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
- f is holomorphic at ∞ $\left(\text{or } f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}\right)$.

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Remark

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Definition

If $c(0) = 0$ in the preceding definition (i.e. f vanishes at ∞), we say that f is a cusp form.

We denote by M_k the space of modular forms of weight k on Γ, and by S_k that of cusp forms.

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Theorem

Let $f \in S_k$ with $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$. Then the Fourier coefficients $a(n)$ of f satisfy

$$a(n) = O\left(n^{\frac{k}{2}}\right).$$

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Corollary

If $k < 0$ and $f \in S_k$, then $f \equiv 0$.

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Definition

For a fixed integer k and any $n = 1, 2, ...$, the operator T_n is defined on M_k by the equation

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

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Observe that writing $n = ad$ and letting $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, we can write

$$(T_n f)(z) = n^{k-1} \sum_{\substack{a \geq 1, ad = n \\ 0 \leq b < d}} d^{-k} f(Az) = \frac{1}{n} \sum_{\substack{a \geq 1, ad = n \\ 0 \leq b < d}} a^k f(Az).$$

Theorem

If f has the Fourier expansion at ∞

$$f = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$$

then

$$T_n f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi i m z}$$

where

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

Theorem

If $f \in M_k$ *and* $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, *then*

$$T_n f(Vz) = (\gamma z + \delta)^k T_n f(z).$$

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Corollary

If $f \in M_k$ then $T_n f \in M_k$. Moreover, if f is a cusp form, then $T_n f$ is also a cusp form.

Definition

A non-zero function f satisfying a relation of the form

$$T_n f = \lambda(n) f$$

for some complex scalar $\lambda(n)$ is called an eigenform of the operator T_n. Moreover, if f is an eigenform for every Hecke operator $T_n, n \geq 1$, then f is called a simultaneous eigenform. A simultaneous eigenform is said to be normalized if $c(1) = 1$, where $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi i m z}$.

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Theorem

Let k be an even integer and $0 \neq f \in S_k$ with $f(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}$. Then f is a normalized simultaneous eigenform if and only if

$$c(m)c(n) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

for all $m, n \geq 1$.

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Definition

If $f(z) = c(0) + \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$, we define the Dirichlet L-function of f as

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Proposition

If $f \in S_k$, then its L-function $L(f,s)$ converges absolutely for $\Re(s) > 1 + \frac{k}{2}$.

Theorem

If f is a normalized Hecke eigenform, then

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

Definition

For $f \in S_k$, define the completed L-function $\Lambda(f, s)$ of f by taking the Mellin transform of f along the upper imaginary axis i.e.

$$\Lambda(f, s) = \int_0^\infty f(iy) y^{s-1}\, dy.$$

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Theorem

We have

$$\Lambda(f,s) = \frac{\Gamma(s)}{(2\pi)^s}L(f,s)$$

for $\Re(s) > 1 + \frac{k}{2}$, *where* $\Gamma(s) = \int_0^\infty e^{-y}y^{s-1}\,dy$ *is the Euler gamma function.*

Theorem

$\Lambda(f,s)$ extends holomorphically to the complex plane and satisfies the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$$

for all $s \in \mathbb{C}$, where $\epsilon(f) = \pm 1$.

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$L(f,s)$ extends to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation

$$\frac{(2\pi)^{k-s}}{\Gamma(k-s)}L(f,s) = i^k\frac{(2\pi)^s}{\Gamma(s)}L(f,k-s)$$

for all $s \in \mathbb{C}$.

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For $X \in \mathbb{C}$ and a cusp form $f \in S_k$ we define the period polynomial of f by the integral transformation

$$r_f(X) = \int_0^{i\infty} (z - X)^{k-2} f(z)\, dz.$$

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Theorem

For $f \in S_k$ and $X \in \mathbb{C}$ we have

$$\begin{aligned} r_f(X) &= \sum_{n=0}^{k-2} \binom{k-2}{n} (-X)^{k-n-2} i^{n+1} \Lambda(f, n+1) \\ &= -\sum_{n=0}^{k-2} \binom{k-2}{n} X^l (-i)^{k-n-1} \Lambda(f, k-n-1). \end{aligned}$$

Corollary

For $f \in S_k$ and $X \in \mathbb{C}$ we have

$$r_f(X) = -\sum_{n=0}^{k-2} \frac{(k-2)!}{n!} \frac{L(f, k-n-1)}{(2\pi i)^{k-n-1}} X^n.$$

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Let $f \in S_k$ and $X \in \mathbb{C}$. Then the period polynomial of f satisfies

$$r_f(X) = -i^k \epsilon(f) X^{k-2} r_f\left(-\frac{1}{X}\right).$$

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This "self-inversive" property of the period polynomial, shows that if ρ is a zero of $r_f(X)$ then so is $-\frac{1}{\rho}$; and so the unit circle is a natural line of symmetry for the period polynomials.

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A polynomial $P(z) = \sum_{i=0}^{d} c_i z^i$ of degree d is said to be *self-inversive* if it satisfies

$$P(z) = \epsilon z^d \bar{P}\left(\frac{1}{z}\right)$$

for some constant ϵ of modulus 1, where $\bar{P}(z) := \sum_{i=0}^{d} \bar{c}_i z^i$ and the bar denotes complex conjugation.

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Lemma

Let $h(z)$ be a nonzero polynomial of degree n with all its zeros in $|z| \leq 1$. Then for $d \geq n$ and any λ with $|\lambda| = 1$, the self-inversive polynomial

$$P^{\{\lambda\}}(z) = z^{d-n} h(z) + \lambda z^n \bar{h}\left(\frac{1}{z}\right)$$

has all its zeros on the unit circle.

For $w = k - 2 \in 2\mathbb{N}$, we have

$$r_f(X) = -\frac{w!}{(2\pi i)^{w+1}} \sum_{n=0}^{w} L(f, w - n + 1) \frac{(2\pi i X)^n}{n!}.$$

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For convenience, we consider the polynomial with real coefficients

$$p_f(X) = -\frac{(2\pi i)^{w+1}}{w!} r_f\left(\frac{X}{i}\right) = \sum_{n=0}^{w} L(f, w-n+1) \frac{(2\pi X)^n}{n!}.$$

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Proposition

$p_f(X)$ is self-inversive and can be written as

$$p_f(X) = q_f(X) + i^k X^w q_f\left(\frac{1}{X}\right)$$

where

$$q_f(X) = \sum_{n=0}^{\frac{w}{2}-1} L(f, w-n+1) \frac{(2\pi X)^n}{n!} + \frac{1}{2} L(f, k/2) \frac{(2\pi X)^{w/2}}{(w/2)!}.$$

Therefore, $r_f(X)$ would have all its zeros on $|z| = 1$ if and only if $q_f(X)$ has all its zeros in $|z| \leq 1$.

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Lemma

Let $f \in S_k$ be a normalized Hecke eigenform and let $L(f,s)$ be its associated L-function. Then, for $s \geq 3k/4$, we have

$$|L(f,s) - 1| \leq 5 \times 2^{-k/4}$$

and, for $s \geq k/2$, we have

$$L(f,s) \leq 1 + 4\sqrt{k}\log(2k).$$

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Proof.

Put $m = k/2 - 1 = w/2$, then for k large enough and $|X| = 1$

$$|q_f(X) - H_m(X)| < |H_m(X)|$$

It follows from Rouché's theorem that $q_f(X)$ has the same number of zeros as $H_m(X)$ inside the unit circle. □

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The principle subgroup of $SL_2(\mathbb{Z})$ of level $N \in \mathbb{N}$ is given by

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We are interested in the congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$

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We denote by $M_k(\Gamma_0(N))$ the space of modular forms of weight k and level N.

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A form $f \in S_k(\Gamma_0(N))$ is a newform if it is a normalized eigenform which cannot be constructed from modular forms of lower levels M dividing N. The other forms are called oldforms. These oldforms can be constructed using the following observations: if $M \mid N$ then $\Gamma_0(N) \subset \Gamma_0(M)$ giving a reverse inclusion of modular forms $M_k(\Gamma_0(M)) \subset M_k(\Gamma_0(N))$.
The space of newforms of level N is denoted by $S_k^{\text{new}}(\Gamma_0(N))$.

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where $\mathbf{1}_N(p)$ is 1 when $p \nmid N$ and is 0 when $p \mid N$.

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where $\mathbf{1}_N(p)$ is 1 when $p \nmid N$ and is 0 when $p \mid N$.
Its completed L-function is defined by

$$\Lambda(f,s) = N^{s/2} \int_0^{\infty} f(iy) y^{s-1}\, dy$$

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$$L(f,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p \text{ prime}} (1 - a(p)p^{-s} + \mathbf{1}_N(p)p^{k-1-2s})^{-1}$$

where $\mathbf{1}_N(p)$ is 1 when $p \nmid N$ and is 0 when $p \mid N$.
Its completed L-function is defined by

$$\Lambda(f,s) = N^{s/2} \int_0^{\infty} f(iy) y^{s-1}\, dy$$

satisfying, as before,

$$\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f,s)$$

and the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s),$$

with $\epsilon(f) = \pm 1$.

The period polynomial associated to f is the degree $k-2$ polynomial

$$r_f(z) = \int_0^{i\infty} f(\tau)(\tau - z)^{k-2}\, d\tau.$$

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The period polynomial of f satisfies

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Corollary

The period polynomial of f further satisfies

$$r_f(z) = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi iz)^n}{n!} L(f, k-n-1).$$

Zeros of Period Polynomials

Zeros of Period Polynomials

For $f \in S_k^{\text{new}}(\Gamma_0(N))$, put $m = \frac{k-2}{2}$ and define

$$P_f(z) = \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2}+j)z^j.$$

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Proposition

The period polynomial of f satisfies

$$r_f\left(\frac{z}{i\sqrt{N}}\right) = i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m\Big(P_f(z) + \epsilon(f)P_f\Big(\frac{1}{z}\Big)\Big).$$

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Therefore, $r_f(z)$ would have all its zeros on $|z| = 1/\sqrt{N}$ if and only if $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.

Lemma

Let $f \in S_k^{new}(\Gamma_0(N))$*. Then the function* $\Lambda(f,s)$ *is monotone increasing for* $s \geq \frac{k}{2}+\frac{1}{2}$*. Moreover, we have*

$$0 \leq \Lambda(f,\frac{k}{2}) \leq \Lambda(f,\frac{k}{2}+1) \leq \Lambda(f,\frac{k}{2}+2) \leq \dots$$

If $\epsilon(f)=-1$*, then* $\Lambda(f,\frac{k}{2})=0$ *and*

$$0 \leq \Lambda(f,\frac{k}{2}+1) \leq \frac{1}{2}\Lambda(f,\frac{k}{2}+2) \leq \frac{1}{3}\Lambda(f,\frac{k}{2}+3) \leq \dots$$

Lemma

Let $f \in S_k^{new}(\Gamma_0(N))$. Then the function $\Lambda(f,s)$ is monotone increasing for $s \geq \frac{k}{2} + \frac{1}{2}$. Moreover, we have

$$0 \leq \Lambda(f, \frac{k}{2}) \leq \Lambda(f, \frac{k}{2} + 1) \leq \Lambda(f, \frac{k}{2} + 2) \leq \ldots$$

If $\epsilon(f) = -1$, then $\Lambda(f, \frac{k}{2}) = 0$ and

$$0 \leq \Lambda(f, \frac{k}{2} + 1) \leq \frac{1}{2}\Lambda(f, \frac{k}{2} + 2) \leq \frac{1}{3}\Lambda(f, \frac{k}{2} + 3) \leq \ldots$$

Proof.

We can write

$$\Lambda(f,s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

where the product is over all the zeros of $\Lambda(f,s)$. □

Lemma

If $f \in S_k^{new}(\Gamma_0(N))$ *and* $0 < a \leq b$*, then*

$$\frac{L(f, \frac{k+1}{2} + a)}{L(f, \frac{k+1}{2} + b)} \leq \frac{\zeta(1+a)^2}{\zeta(1+b)^2}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ *is the Riemann zeta function.*

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Proof.

We have that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

and

$$-\frac{L'}{L}(f, s) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}$$

where $|\Lambda_f(n)| \leq 2n^{\frac{k-1}{2}} \Lambda(n)$. □

Theorem

For $k = 4$, $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.

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Proof.

Here $m = (k-2)/2 = 1$, so $P_f(z) = \Lambda(f,2) + \Lambda(f,3)z$.
If $\epsilon(f) = -1$, then the roots of $P_f(z) - P_f(1/z) = \Lambda(f,3)(z - 1/z)$ are at $z = \pm 1$, which lie on the unit circle.
If $\epsilon(f) = 1$, then for $z = e^{i\theta}$ on the unit circle, $P_f(z) + P_f(1/z) = 2\Lambda(f,2) + \Lambda(f,3)(e^{i\theta} + e^{-i\theta}) = 2\Lambda(f,2) + 2\Lambda(f,3)\cos(\theta)$, which vanishes when $\cos(\theta) = -\Lambda(f,2)/\Lambda(f,3)$. But, $\Lambda(f,2) < \Lambda(f,3)$, and so the equation has two solutions for $\theta \in [0, 2\pi)$. □

Theorem

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Proof.

If $\epsilon(f) = -1$, we do the same as above.
If $\epsilon(f) = 1$, letting $z = e^{i\theta}$ we have

$$P_f(z) + P_f\left(\frac{1}{z}\right) = 6\Lambda(f,3) + 8\Lambda(f,4)\cos\theta + 2\Lambda(f,5)\cos 2\theta.$$

We want to show this has two zeros in $[0,\pi)$ and thus four zeros in $[0,2\pi)$. Note that

$$\frac{d}{d\theta}\left[P_f(e^{i\theta}) + P_f(e^{-i\theta})\right] = -8\sin\theta(\Lambda(f,4) + \Lambda(f,5)\cos\theta),$$

we have critical points at $0,\pi$ and the solution $\theta_0 \in [0,\pi)$ to $\cos\theta = -\frac{\Lambda(f,4)}{\Lambda(f,5)}$. □

Proof.

To get two roots in $[0, \pi)$ we need $P_f(e^{i\theta}) + P_f(e^{-i\theta})$ to be positive at $\theta = 0$ and π and negative at $\theta = \theta_0$. At $\theta = 0,\ P_f(e^{i\theta}) + P_f(e^{-i\theta}) = 6\Lambda(f,3) + 8\Lambda(f,4) + 2\Lambda(f,5) > 0$. Positivity at $\theta = \pi$ is equivalent to

$$\Lambda(f,5) + 3\Lambda(f,3) + > 4\Lambda(f,4)$$

while negativity at $\theta = \theta_0$ is equivalent to

$$\Lambda(f,5)^2 + 2\Lambda(f,4)^2 < 3\Lambda(f,3)\Lambda(f,5).$$

We show these inequalities using a clever application of the Hadamard formula from earlier. □

We have for $z = e^{i\theta}$

$$P_f(z) + P_f\Big(\frac{1}{z}\Big) = \binom{2m}{m}\Lambda(f, \frac{k}{2}) + 2\sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2}+j)\cos(j\theta),$$

and

$$P_f(z) - P_f\Big(\frac{1}{z}\Big) = 2\sum_{j=1}^{m}\binom{2m}{m+j}\Lambda(f, \frac{k}{2}+j)\sin(j\theta).$$

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For $8 \leq k \leq 14$, $P_f(z) + \epsilon(f)P_f(1/z)$ has all its zeros on the unit circle.

Proof.

Using classical work of Pólya and Szegö on trigonometric polynomials, together with our lemmas, the result is true if

$$N \geq \max_{1\leq j\leq k/2-2}\left(\frac{2\pi}{k/2-j-1}\right)^2\frac{\zeta(j+1/2)^4}{\zeta(j+3/2)^4}.$$

□

Proof.

For any given k, we can compute this bound. Thus, for $k = 8$ it suffices to take $N \geq 142$; for $k = 10$ it suffices to have $N \geq 64$; for $k = 12$ it suffices to have $N \geq 45$; for $k = 14$ it suffices to have $N \geq 42$. We can use PARI to check for those newforms not covered by this bound for weights $8 \leq k \leq 14$. □

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Remark

Eventually, the inequality cannot furnish a bound better than $4\pi^2$ for N, and so we must turn to another approach for large k and small N.

Proposition

$P_f(z)$ can be written as

$$P_f(z) = (2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} L(f, 2m+1)Q_f(z)$$

where

$$Q_f(z) = z^m \sum_{j=0}^{m-1} \frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j \frac{L(f, 2m+1-j)}{L(f, 2m+1)} + \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)}.$$

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$$+ \frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda(f, \frac{k}{2})}{L(f, 2m+1)}.$$

Therefore, we need to study the zeros of

$$\left(Q_f(z) + \epsilon(f)Q_f\left(\frac{1}{z}\right)\right).$$

But, note that

$$Q_f(z) - Q_f\left(\frac{1}{z}\right) = 2\Im\Big(Q_f(z)\Big)$$

and

$$Q_f(z) + Q_f\left(\frac{1}{z}\right) = 2\Re\Big(Q_f(z)\Big).$$

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and

$$Q_f(z) + Q_f\left(\frac{1}{z}\right) = 2\Re\Big(Q_f(z)\Big).$$

Theorem

For $k \geq 16$, the real and imaginary parts of $Q_f(z)$ have all their zeros on the unit circle.

Algebraic Detour

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Let $\mathbb{Q} \subset K \subset \mathbb{C}$ be a field.
We can consider K as a vector space over $\mathbb{Q}$.

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Let $\mathbb{Q} \subset K \subset \mathbb{C}$ be a field.
We can consider K as a vector space over $\mathbb{Q}$.
K is called an algebraic number field if the dimension of this vector space is finite. This dimension is called the degree of K.
The smallest K which contains a is denoted by

$$K = \mathbb{Q}(a).$$

An embedding of a number field K in $\mathbb{C}$ is an injective field homomorphism of K into $\mathbb{C}$.

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Let K be a number field of degree n. Then there are exactly n different embeddings of K in $\mathbb{C}$.

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We usually arrange the embeddings in a certain order and denote them by

$$K \to K^{(j)} \subset \mathbb{C}$$
$$a \to a^{(j)},\ j = 1, \ldots, n.$$

We put the n embeddings together into a single $\mathbb{Q}$-linear injective mapping

$$K \to \mathbb{C}^n,\ \ a \to (a^{(1)}, a^{(2)}, \ldots, a^{(n)}).$$

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An embedding is called real if its image is contained in $\mathbb{R}$. K is called totally real if it admits only real embeddings.

The *trace* and *norm* of an element $a \in K$ over $\mathbb{Q}$ are given, respectively, by

$$Tr(a) = Tr_{K/\mathbb{Q}}(a) = \sum_{j=1}^{n} a^{(j)}, \quad N(a) = N_{K/\mathbb{Q}}(a) = \prod_{j=1}^{n} a^{(j)}.$$

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Definition

Let K be an algebraic number field. The ring of integers of K is defined as

$$\mathcal{O}_K = K \cap \overline{\mathbb{Z}},$$

where $\overline{\mathbb{Z}}$ is the algebraic closure of $\mathbb{Z}$.

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Theorem

Let K be a number field of degree n. Then $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.

Write $\mathcal{O}_K = \langle a_1, a_2, \ldots, a_n \rangle_{\mathbb{Z}}$ and let

$$A = \begin{pmatrix} a_1^{(1)} & a_1^{(2)} & \ldots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & \ldots & a_2^{(n)} \\ . & . & & . \\ . & . & & . \\ a_n^{(1)} & a_n^{(2)} & \ldots & a_n^{(n)} \end{pmatrix}.$$

Then the *discriminant* D_K of K is given by $D_K = (\mathrm{detA})^2$.

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We define the *Norm* of an integral ideal $\mathfrak{a}$ as

$$N(\mathfrak{a}) := |\mathcal{O}_K/\mathfrak{a}|.$$

Hilbert Modular Forms

Let K be a totally real number field of degree n.

Hilbert Modular Forms

Let K be a totally real number field of degree n.
If we attach to the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

the tuple $(M_1, \ldots, M_n)$ where

$$M_j = \begin{pmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{pmatrix}, \ j = 1, \ldots, n$$

we obtain an embedding of groups

$$GL_2(K) \hookrightarrow GL_2(\mathbb{R})^n.$$

The group

$$GL_2^+(K) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K) : \det\gamma_j > 0 \ \text{ for } \ j = 1, \ldots, n \right\}$$

acts on $\mathbb{H}^n$ by coordinate linear fractional transformations, i.e. for $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$

$$z \to \gamma z = (\gamma_i z_i)_i = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \ldots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right).$$

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We define the *full Hilbert modular group* to be

$$\Gamma_K := GL_2^+(\mathcal{O}_K).$$

Definition

A holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ is called a holomorphic Hilbert modular form of weight $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ for Γ_K, if for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$

$$f(\gamma z) = \prod_{i=1}^{n} \det(\gamma_i)^{-k_i/2} \left(c^{(i)} z_i + d^{(i)}\right)^{k_i} f(z).$$

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If $k_1 = k_2 = \cdots = k_n := k$ then f is said to have parallel weight, and is simply called a holomorphic Hilbert modular form of weight $k \in \mathbb{Z}$. We denote the space holomorphic Hilbert modular forms of weight k on Γ_K by $M_k(\Gamma_K)$. Moreover, If $f \in M_k(\Gamma_K)$ vanishes at the cusps of Γ_K, we call it a cusp form and denote this space by $S_k(\Gamma_K)$ as usual.

$f \in S_k(\Gamma_K)$ has an associated L-function given by

$$L(f, s) := \sum_{\substack{\mathfrak{n} \in \mathcal{O}_K \\ \mathfrak{n} \neq 0}} \frac{a(\mathfrak{n})}{N(\mathfrak{n})^s}.$$

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If $U = (\mathcal{O}_K^*)_+$ then letting $f(z) = f(z_1, \ldots, z_n)$, $N(z) = z_1 \ldots z_n$ and $dz = dz_1 \ldots dz_n$, we can define the completed L-function by

$$\Lambda(f,s) := \int_{(\mathbb{R}_+)^n/U} f(iy) N(y)^{s-1}\, dy,$$

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$$\Lambda(f,s) := \int_{(\mathbb{R}_+)^n/U} f(iy) N(y)^{s-1}\, dy,$$

which satisfies

$$\Lambda(f,s) = \left(\frac{D_K}{(2\pi)^n}\right)^s \Gamma(s)^n L(f,s) \tag{1}$$

and the functional equation

$$\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$$

where $\epsilon(f) \in \{\pm 1\}$.

We further define the period polynomial of a parallel weight k Hilbert modular eigenform f as

$$r_f(X) := \int_{i((\mathbb{R}_+)^n/U)} f(\tau)(N(\tau) - X)^{k-2}\, d\tau.$$

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In analogy with the classical case,

Theorem

The period polynomial r_f of f satisfies

$$r_f(X) = \sum_{\ell=0}^{k-2} (-1)^\ell i^{n(k-\ell-1)} \binom{k-2}{\ell} X^\ell \Lambda(f, k-\ell-1).$$

Zeros of Period Polynomials

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let K be a number field of degree n and f be a parallel weight k Hilbert modular eigenform.

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let K be a number field of degree n and f be a parallel weight k Hilbert modular eigenform. Put $m := \frac{k-2}{2}$ and define

$$P_f(X) = \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda\left(f, \frac{k}{2}+j\right)X^j$$

and

$$Q_f(X) = \frac{1}{\Lambda(f, 2m+1)}P_f(X).$$

Zeros of Period Polynomials

let K be a number field of degree n and f be a parallel weight k Hilbert modular eigenform. Put $m := \frac{k-2}{2}$ and define

$$P_f(X) = \frac{1}{2}\binom{2m}{m}\Lambda(f, \frac{k}{2}) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda\left(f, \frac{k}{2}+j\right)X^j$$

and

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Proposition

$r_f(i^{n+2}X)$ is self-inversive and can be written as

$$r_f(i^{n+2}X) = i^{n(2m+1)}\epsilon(f)\Lambda(f, 2m+1)X^m\left[Q_f(X) + \epsilon(f)Q_f\left(\frac{1}{X}\right)\right].$$

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Then, $r_f(X)$ would have all its zeros on the unit circle if and only if $Q_f(X)$ has all its zeros inside the unit circle.

Theorem

For $k = 4$ and $k = 6$, $P_f(X) + \epsilon(f) P_f(1/X)$ has all its zeros on the unit circle.

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For large weights, we compare $Q_f(X)$ to X^m and use Rouchés Theorem to show $Q_f(X)$ has all its zeros inside the unit circle.
On $|X| = 1$, we show $|Q_f(X) - X^m| \leq T_n(m)$ where

$$T_n(m) = \frac{1}{2}\frac{\Gamma(m+1)^{n-2}}{\Gamma(2m+1)^{n-1}}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^m\left(\frac{11}{5}\right)^n + \sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{(2\pi)^n(n!)^2}{n^{2n}}\right)^j\left(\frac{\Gamma(2m+1-j)}{\Gamma(2m+1)}\right)^{n-1}\left(\frac{\zeta(1/2+m-j)}{\zeta(1/2+m)}\right)^{2n}$$

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We show that $T_n(m)$ is also decreasing in m. Therefore, once we have $T_2(m_0) < 1$ for some m_0, we then automatically get that $T_n(m) < 1$ for any $n \geq 2$ and $m \geq m_0$. We do this by showing $T_n(m+1) - T_n(m) \leq 0$.

Thank You!