We decompose the Hecke L-series according to the classes C of the ideal class group

$$L(\chi, s) = \sum_{C \in CL(F)} L(C, \chi, s)$$

where

$$L(C, \chi, s) = \sum_{\substack{\mathfrak{a} \in C \\ \mathfrak{a} \text{ integral}}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

and deduce a functional equation for those.

We have the following bijection

$$(\hat{C} \cap \hat{\mathcal{O}})/\mathcal{O}^{\times} \to \{\mathfrak{a} \in C : \mathfrak{a} \; integral\}$$

given by

$$a\mathcal{O}^{\times} \to (a),$$

where $\hat{C} \in \hat{F}^{\times}/F^{\times}$ corresponds to the class $C \in CL(F)$ with respect to the isomorphism

$$CL(F) \simeq \hat{F}^{\times}/F^{\times}$$

given by

$$\mathfrak{b} = \mathfrak{b}_1^{v_1} \dots \mathfrak{b}_r^{v_r} \to \hat{b_1}^{v_1} \dots \hat{b_r}^{v_r}.$$

Using this, and the fact that

$$Nm(\mathfrak{a}) = Nm((a)) = |N(a)|,$$

we can write

$$L(C, \chi, s) = \sum_{a \in R} \frac{\chi((a))}{|N(a)|^s}$$

where R is a system of representatives of

$$(\hat{C} \cap \hat{\mathcal{O}})/\mathcal{O}^\times.$$

We want to write this function as a Mellin Transform to be able to use the following result:

(Neukirch, Theorem (1.4) of Chapter 7)

Theorem

Let $f, g : \mathbb{R}_{>0} \to \mathbb{C}$ be continuous functions such that

$$f(y) = a_0 + O(e^{-cy^\alpha}) \text{ and } g(y) = b_0 + O(e^{-cy^\alpha})$$

as $y \to \infty$, with positive constants c, α. If these functions satisfy the equation

$$f(\frac{1}{y}) = Cy^k g(y)$$

for some $k \in \mathbb{R}_{>0}$ and $C \in \mathbb{C}^\times$, then we get that the integrals $M(f, s)$ and $M(g, s)$ have analytic continuation to $\mathbb{C} \setminus \{0, k\}$, and they satisfy the functional equation

$$M(f, s) = CM(g, k - s).$$

Define for $s \in \mathbb{C}$ the following

$$\Gamma_F(s) = \int_{F_{\mathbb{R},+}} N(e^{-y}y^s)\frac{dy}{y},$$

$$\Lambda(C,\chi,s) = (|d_F|N(\mathfrak{m}))^{s/2}N(\pi^{-s/2})\Gamma_F(s/2)L(C,\chi,s)$$

and

$$\theta(C,\chi,z) = \sum_{a\in\hat{C}\cap\hat{\mathcal{O}}} \chi_f(a)N(a^u)e^{\pi i\langle az/|md|,a\rangle}$$

with $m, d \in \hat{\mathcal{O}}$ s.t. $\mathfrak{m} = (m)$ and $\mathfrak{d} = (d)$.

Theorem

$$\Lambda(C, \chi, s) = M(f, s')$$

where

$$f(t) = f_F(C, \chi, t) = \frac{c(\chi)}{w} \int_B N(x^{(u-iv)/2}) \theta(C, \chi, ixt^{1/n})\, d^*x,$$

w is the number of roots of unity in F, $s' = \frac{1}{2}(s + Tr(u - iv)/n)$, and $c(\chi) = N(|md|^{-u+iv})^{1/2}$.

Define

$$W(\chi) = \left[i^{Tr(\bar{v})} N\left(\left(\frac{md}{|md|} \right) \right)^{\bar{v}} \right]^{-1} \frac{\tau(\chi_f)}{\sqrt{N(\mathfrak{m})}},$$

where

$$\tau(\chi_f) = \sum_{x \pmod{m}} \chi_f(x) e^{2\pi i Tr(x/md)}$$

Notice that $|W(\chi)| = 1$.

Theorem

The function $\Lambda(\chi, s)$ defined for $Re(s) > 1$ admits analytic continuation to

$$\mathbb{C} \setminus \{Tr(-u + iv)/n, 1 + Tr(u + iv)/n\}$$

and satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1 - s).$$

If $\mathfrak{m} \neq 1$ or $u \neq 0$, it is holomorphic on all of $\mathbb{C}$.